REAL VALUE

CHAPTER 4 AMERICAN PERPETUAL REAL OPTIONS

Perhaps the first real American perpetual options were identified by the University of Manchester economics professor, Jevons (1871). He noted that in effect there were real (environmental) options in the prospective use of a commons, which "might be allowed to perish at any moment, without harm, if we could have it re-created with equal ease at a future moment, when need of it arises".

4.1 SAMUELSON-MCKEAN AMERICAN PERPETUAL OPTION VALUE

Samuelson (1965) (with McKean) developed an analytical solution for a perpetual American (that can be exercised any time) option. This is appropriate to value a perpetual opportunity to convert land into buildings. There are five inputs required for this early real call option model: the riskfree interest rate (r), the yield on the built property (δ), the construction cost (K), the value (V) and the expected volatility (σ) of the built property. The value of the real call option is

$$\mathrm{ROV_C} = (V^* - K)\left(\frac{V}{V^*}\right)^{\beta_1} \tag{4.1}$$

where $$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} \tag{4.2}$$

and $$V^* = \frac{\beta_1}{\beta_1 - 1} K. \tag{4.3}$$

When the value of the underlying built property reaches V*, the land conversion option should be exercised, that is construction should commence.

Figure 4.1

	A	B	C
1	**SAMUELSON-MCKEAN: REAL AMERCAN PERPETUAL CALL OPTION**		
2	AFTER SAMUELSON-McKEAN (1965)		
3			
4	INPUTS		
5	Yield, δ	0.04	
6	Riskless rate, r_f	0.04	
7	Asset Volatility	0.20	
8	Investment Cost, K	$1.00	
9	Developed Asset Value, V	$1.00	
10			
11	OUTPUTS		
12	β	2.00	(B5-B6+(B7^2)/2+SQRT((B6-B5-(B7^2)/2)^2+2*B6*(B7^2)))/(B7^2)
13	V*	$2.00	B8*B12/(B12-1)
14	DEVELOPMENT OPTION VALUE	$0.25	IF(B9>$B$13,B15,($B$13-$B$8)*((B9/$B$13)^$B$12))
15	INTRINSIC OPTION VALUE	$0.00	IF(B9-$B$8>0,(B9-$B$8),0)
16	DELTA	0.50	MIN((((B9*($B$12-1))/(($B$8*$B$12)))^($B$12-1)),1)

Figure 4.1 shows both the intrinsic and option value of land which is a development site for an office building with a construction cost of $1 (or $1 per square foot, if you like). The yield of the office when constructed is expected to be 4%, the riskfree interest rate is 4%, and the volatility of the value of the office building is expected to be 20%. Note that under these assumptions, the office should not be built until the building is worth $2.00 (or $2 per square foot). The intrinsic value (V-K) is nil, but the real option to defer construction until V reaches V* is worth $ .25.

Figure 4.2

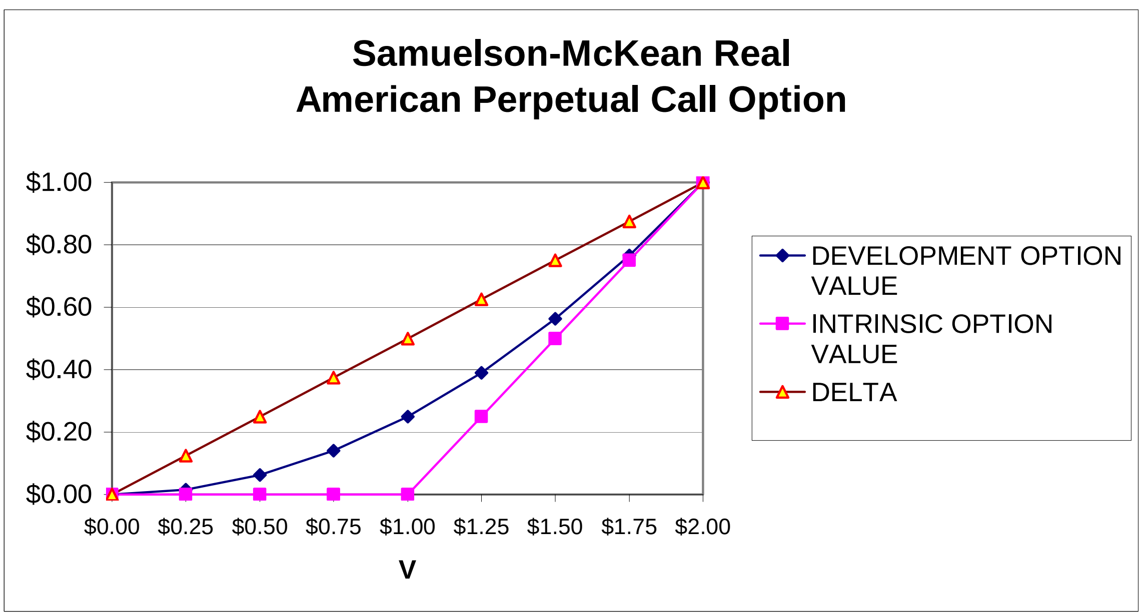


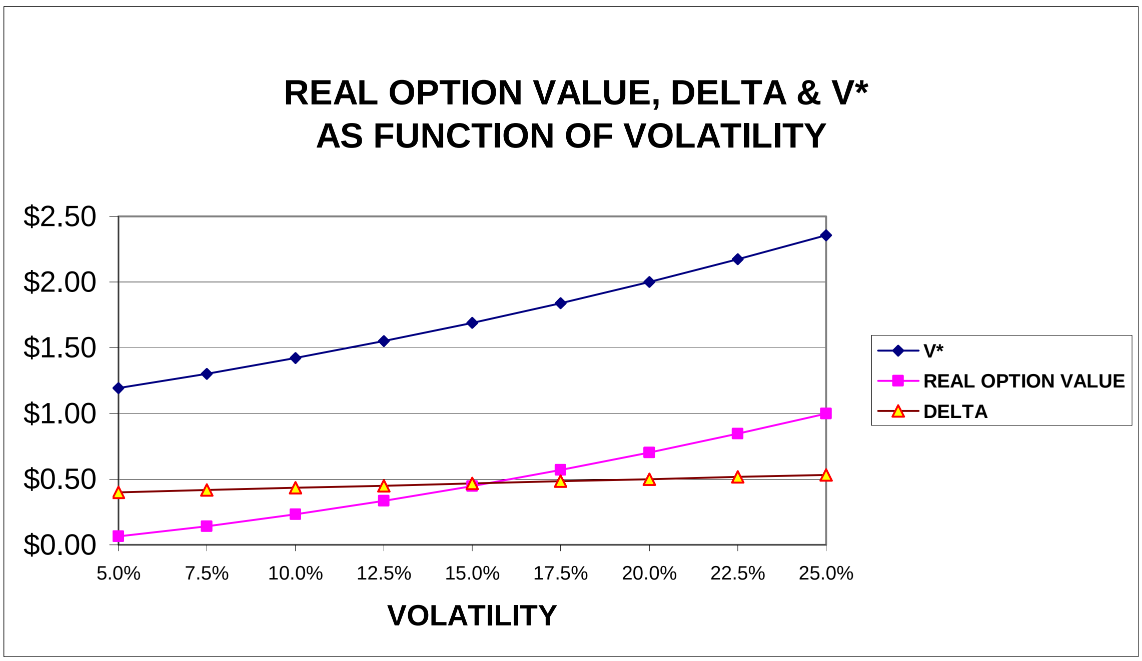
Figure 4.2 shows the real American perpetual call option over a range of V from 0 to \$2.00. There is nil intrinsic option value until V>\$1; the land conversion option is greater than the intrinsic value until V>V*, that is past \$2, when then the option value equals the intrinsic value.

Delta is the rate of change of the call option value as the value of the underlying asset changes (it is also the slope of the real call option curve). The ROV delta is

$$\Delta_c^{ROV} = \left(\frac{V(\beta_1 - 1)}{K\beta_1} \right)^{\beta_1 - 1} \tag{4.4}$$

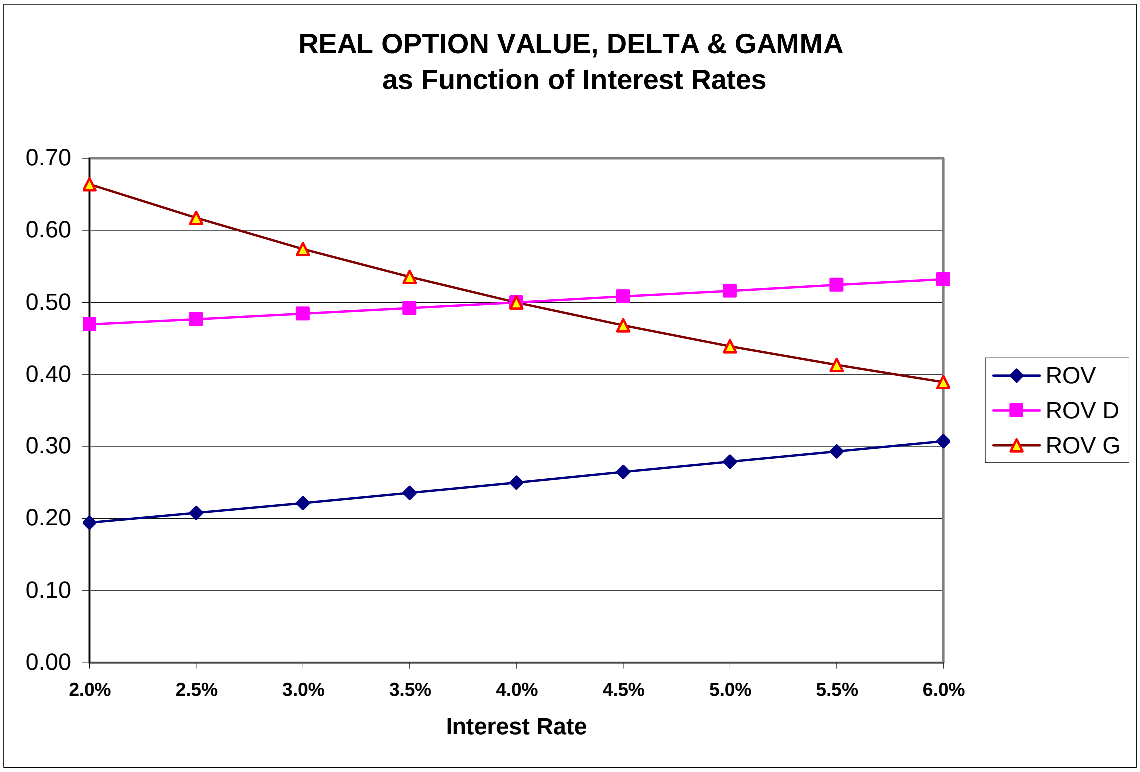
(see Appendix 4A, equation A7), and increases from 0 when V is 0 to 1 when V>V*.

Figure 4.3



A real call option value increases with the value of the underlying asset, and also with increases in the volatility of the underlying asset. Figure 4.3 shows that the real call option value increases with the volatility of the underlying asset, and also that V*, the critical value at which the investment should made, increases with volatility. The assumptions in this illustration are that currently V=K=\$1, that is the call option is at-the-money, and the expected future volatility ranges from 5%, where the call option has low value, to 25% per annum, where the call option to build is 100% of the gross value of the office building when constructed. At 25% volatility, however, the developed property would have to be worth more than \$2.36 before exercising the option. Notice that the Delta increases slightly with the increase of volatility. Gamma (see Appendix 4A, equation A8) is the rate of change of Delta, which is affected by volatility, interest rates and the payout yield. Value, Delta and Gamma as functions of the interest rate level are shown in Figure 4.4.

Figure 4.4



One reason for examining the sensitivities of the real American perpetuity to changes in the parameters is to envisage some of the problems in trying to replicate such an option using dynamic positions in the underlying asset plus positions in other similar real options and financial/commodity securities. Note that the ROV (real option value), ROV Δ (rate of change of ROV as V changes), and ROV Γ (rate of change of delta changes as V changes) are affected by β changes, due to alterations in the interest rate, asset yield or asset volatility. So ROV, ROV Δ and ROV Γ are all likely to change over time, as the underlying asset and other economic parameters change.

4.2 DERIVATION OF THE AMERICAN PERPETUAL OPTION MODEL

Samuelson (1965) was actually concerned with valuing an American perpetual warrant (presumably on a traded stock), but subsequently several authors including Tourinho (1979), Brennan and Schwartz (1985) and McDonald and Siegel (1986) extended this model to cover non-traded real assets and the optimal timing of investing in an irreversible project.

Consider a company with the opportunity to invest in a certain project; the investment cost is irreversible or irrecoverable once incurred. The investment cost K is known, or deterministic, while the present value of the project's gross cash flows, V, follows a geometric Brownian motion:

$$dV = \mu V dt + \sigma V dt \tag{4.5}$$

where μ is the growth rate or drift parameter; σ the volatility and dz the increment of a Wiener process. In a risk neutral world, or in perfect hedging which earns the riskless return, μ=r-δ, where δ is the asset yield. Since the company has the right, but not the obligation, to invest in such a project, the investment opportunity can be seen as a call option. that explains the movements on the value of the opportunity to invest. Appendix 4A shows that the value of the opportunity to invest can be represented by the following differential equation:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} + (r-\delta)V\frac{\partial F}{\partial V} - rF = 0 \tag{4.6}$$

Equation (4.6) is a second order linear ordinary differential equation ("ODE") with the solution:

$$F(V) = AV^{\beta_1} + BV^{\beta_2} \tag{4.7}$$

where roots β_1 and β_2 are the solutions of the characteristic quadratic equation[1]:

$$\frac{1}{2}\sigma^2\beta(\beta-1)+(r-\delta)\beta-r=0 \tag{4.8}$$

and therefore β_1 and β_2 are respectively:

$$\beta_1=\frac{1}{2}-\frac{(r-\delta)}{\sigma^2}+\sqrt{\left[\frac{(r-\delta)}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2r}{\sigma^2}}>1 \tag{4.9}$$

and

$$\beta_2=\frac{1}{2}-\frac{(r-\delta)}{\sigma^2}-\sqrt{\left[\frac{(r-\delta)}{\sigma^2}-\frac{1}{2}\right]^2+\frac{2r}{\sigma^2}}<0 \tag{4.10}$$

Equation (4.7) gives the value of the option to invest in a project with a gross present value V. As V decreases, the value of the option to invest in the project has also to decrease and therefore the value yielded by equation (4.7) has to decrease. Moreover, due to the stochastic process followed by V, when V reaches zero it will stay there forever. This boundary condition implies that B has to be zero. Therefore the solution can be written as:

$$F(V)=AV^{\beta_1} \tag{4.11}$$

The value of the arbitrary constant, A, is found subjecting equation (4.11) to two boundary conditions: the value matching and the smooth pasting conditions. The value matching condition states that when V reaches a trigger value, V*, the option will be exercised and therefore, at that point, the investor receives the net present value of the project. In other words, at V* the value of the option to invest equals the net present value of the investment. In mathematical terms:

[1] See Appendix 4A for the basic solution of a quadratic equation.

$$AV^{*\beta_1} = V^* - K \tag{4.12}$$

The smooth pasting condition states that at the trigger value the derivatives of the two functions in (4.12) must be equal. Notice that the two functions in equation (4.12), the option and the net present value of the investment, meet tangentially at the trigger point.

$$\beta_1 AV^{*\beta_1 - 1} = 1 \tag{4.13}$$

Solving (4.12) and (4.13) we obtain the arbitrary constant and the trigger function[2]:

$$A = [\frac{(\beta_1 - 1)^{\beta_1 - 1}}{\beta_1^{\beta_1} K^{\beta_1 - 1}}] = \frac{K}{(\beta_1 - 1)V^{*\beta_1}} = \frac{V^* - K}{V^{*\beta_1}} \tag{4.14}$$

$$V^* = \frac{\beta_1}{\beta_1 - 1} K \tag{4.15}$$

Finally, substituting for A in equation (4.11), we obtain the value of the option to invest in a project where the present value of future cash flows follows a geometric Brownian motion:

$$F(V) = \frac{K}{\beta_1 - 1}\left(\frac{V}{V^*}\right)^{\beta_1} = (V^* - K)\left(\frac{V}{V^*}\right)^{\beta_1} \tag{4.16}$$

The spreadsheet “proof” that this is the solution of equation 4.6 is shown in Figure 4.5.

[2] See Appendix 4A.

Figure 4.5

	A	B	C
1	**Perpetual American Option**		
2	INPUT		
3	V	100.00	
4	K	100.00	
5	σ	0.20	
6	r	0.04	
7	δ	0.04	
8	OUTPUT		
9	F(V)	25.00	IF(B3<B12,B13*(B3^B14),B10)
10	V-K	0.00	B3-B4
11	F'(V)	0.50	IF(B3<B12,B13*B14*(B3^(B14-1)),1)
12	V*	200.00	(B14/(B14-1))*B4
13	A	0.0025	(B12-B4)/(B12^B14)
14	β_1	2.00	
15	β_1	0.5-(B6-B7)/(B5^2)+SQRT(((B6-B7)/(B5^2)-0.5)^2 + 2*B6/(B5^2))	
16	ODE	0.00	0.5*(B5^2)*(B3^2)*B17+(B6-B7)*B3*B11-B6*B9
17	F''(V)	0.01	IF(B3<B12,B13*B14*(B14-1)*(B3^(B14-2)),0)
18	F'(V*)	1.00	B13*B14*(B12^(B14-1))
19	F(V*)	100.00	IF(B3<B12,B13*(B12^B14),B10)
20	V*-K	100.00	B12-B4

F(V) is from equation 4.16, F'(V) and F''(V) are from Appendix 4A (A7, A8); the calculation for F(V) and its first and second derivative is substituted into equation 4.6, row 16 ODE, which is equal to zero.

4.3 INVESTMENT TIMING and SENSITIVITIES

The trigger value function presented in equation (4.15) implies an investment rule different from the classical theories. The Marshall (1890) rule is to invest as soon as the present value of the expected cash flows equals the investment cost, in other words the optimal Marshallian investment time is given by:

$$V^* = K \tag{4.17}$$

The trigger function as defined by equation (4.15) is larger than the Marshallian trigger by $\beta_1/(\beta_1-1)$. Since $\beta_1>1$, the real option $V^*>K$. With the parameters in Figure 4.1, $V^*=2K$, since $\beta_1=2$. As investment cost increases, the value of the Marshallian and the trigger function also increase. The two functions are further apart for higher values of investment cost (since K is multiplied by a number greater than 1). Thus using a real options investment model is a conservative view of the investment, recommending greater caution than traditional valuation methods.

The Marshallian trigger is indifferent to volatility. In the Marshallian theory the future investment cost and the present value of the future cash flows are not stochastic. Real options have an entirely different approach, the future is stochastic and therefore uncertainty should be considered in the valuation framework. As uncertainty increases, the trigger function increases. Thus, in more volatile markets the investors should wait longer to invest, with the expectation that some of the uncertainty will be resolved during the waiting process and/or V will increase sufficiently to justify investment.

We indicated in the first section that V*, the ROV, ROV delta (Δ) and ROV gamma (Γ) are all sensitive to changes in interest rates, asset volatility and asset yield. Conventional traded option pricing texts refer to the interest rate sensitivity of the real option as “rho” and the volatility sensitivity as “vega”. Calculation of these sensitivities (option “Greeks”) for real American perpetual options is somewhat complicated, since interest rates (and asset yields) appear inside the square root solution of β_1, and volatility is a denominator and also inside the square root solution. However, the intuition consistent with Figures 4.3 and 4.4 is that V*, ROV and ROV Δ are positive functions of both the interest rate and volatility, and negative functions of asset yield, while ROV Γ has the opposite sensitivities.

It is assumed in the derivation of real option pricing model that a hedged portfolio of an option and a short position in the underlying assets can be constructed so that over time its return is riskless (see equations A1 through A6). Perpetual American options deltas and gammas are somewhat different than for finite European options, as shown in Appendix 4A. Such discrete time hedges are likely to result in marginal gains/losses, even without transaction costs, as shown in Figure 4.6.

Figure 4.6

	A	B	C	D	E	F	G
1	**Hedging a Perpetual American Option**						
2	YEAR	2006	2007	2008	2009	2010	
3	**V**	**100.00**	**110.00**	**99.00**	**107.91**	**100.00**	
4	K	100.00	100.00	100.00	100.00	100.00	
5	σ	0.20	0.20	0.20	0.20	0.20	
6	r	0.04	0.04	0.04	0.04	0.04	
7	δ	0.04	0.04	0.04	0.04	0.04	
8	OUTPUT						
9	ROV	25.00	30.25	24.50	29.11	25.00	
10	HEDGE	-50.00	-44.50	-49.95	-45.14	-49.09	
11	ALTER HEDGE		5.50	-5.45	4.81	-3.95	
12	GAIN ROV		5.25	-5.75	4.61	-4.11	
13	GAIN V		-5.00	4.45	-4.50	3.31	
14	NET GAIN		0.25	-1.30	0.11	-0.80	
15	**CUM GAIN**		**0.25**	**-1.05**	**-0.93**	**-1.74**	
16	F'(V)	0.50	0.55	0.50	0.54	0.50	
17	V*	200.00	200.00	200.00	200.00	200.00	
18	A	0.0025	0.0025	0.0025	0.0025	0.0025	
19	β_1	2.00	2.00	2.00	2.00	2.00	
20	F''(V)	0.01	0.01	0.01	0.01	0.01	
21	V-K	0.00	10.00	-1.00	7.91	0.00	
22	SUPPOSE V IS TIMES		**1.10**	**0.90**	**1.09**	**0.93**	
23	HEDGE	-B16*B3	B10+C11				
24	ALTER HEDGE		(C16-B16)*C3				
25	GAIN ROV		-B9+C9				
26	GAIN V		-(B10/B3)*(-C3+B3)				
27	NET GAIN		C12+C13				
28	**CUM GAIN**		**SUM($C$14:C14)**				
29							
30	LAND_INVEST HOLDS A PERPETUAL AMERICAN REAL CALL OPTION						
31	HEDGE BY SHORTING DELTA OF V BEGINNING OF EACH YEAR.						
32							RISK
33	RISK ROV		21.00%	-22.99%	18.44%	-16.45%	22.95%
34	RISK ROV HEDGED		1.00%	-5.19%	0.46%	-3.21%	2.97%

Over this particular time sequence of changes in V (shown in bold blue) and parameters, the "delta" hedging of a long position in the real option by a short Δ times V position in an identical underlying asset, assuming no transaction costs, and readjusting the short position every end period, results in gain/losses for every period (shown in bold red). The RISK ROV is the change in the ROV divided by the initial ROV; the RISK ROV HEDGED is the change in the ROV less the change in the hedge gain/loss, divided by the initial ROV. The last column RISK is the standard deviation of the respective rows, showing that although delta hedging is not perfect, even without transaction costs, the risk reduction achieved through delta hedging is significant. This illustrates the opportunities/problems in creating a synthetic real option by dynamic trading in the underlying assets such as commodity futures or asset-related shares or trusts.

SUMMARY

This chapter presents the real perpetual American option model, extended to investment decisions. The main idea is that an investor does not maximise the value of his investment decision entering the market when the present value of the cash flows equals the investment cost. The original Samuelson-McKean (1965) model is shown, along with indications of ROV sensitivities. Then the ROV and V* are derived as the solution to a ordinary differential equation. It is easy, especially in Excel, to show that the solution, along with the first and second derivatives of the ROV, actually solves the fundamental equation. Along those lines, it is not hard to imagine the complexities that might be encountered in trying to replicate with a "low tracking error" any ROV along a time frame, by altering appropriate positions directly in the underlying asset (or similar securities), other real options, and financial and commodity forwards/futures.

EXERCISES

EXERCISE 4.1 An office building of 100,000 square feet in Manchester would be worth £500 per square foot, and costs £450 per square foot to build (including "amenities"). MBA Build has received planning permission on a suitable plot of land which it wants to acquire for the development. The volatility of office buildings is 20%, interest rates 4%, expected payout 4%. What is the value of this land? At what building value should MBA Build start the development?

EXERCISE 4.2 A small house of 2,000 square feet in London would be worth £750 per square foot, and costs £850 per square foot to build, including land costs. City Ltd. has identified a suitable plot of land with planning permission, which it wants to acquire for the development. The volatility of London houses is 50%, interest rates 4%, expected payout 4%. What is the value of this development? At what house value should City start the development? Suppose City had bank deposits of £550,000, liabilities of £500,000, and is able to acquire this development for £200,000. What is the company worth per share, with 1,000,000 outstanding shares?

EXERCISE 4.3 UMAN has the right in perpetuity to acquire UMIT which is now worth £1.1 billion for £1.7 billion. What is the value of the option, and the optimal value at which the acquisition should be made if the volatility of UMIT is 20%, expected payout is 4%, same as the current interest rate? What are the problems in applying real option theory to this acquisition?

PROBLEMS

PROBLEM 4.4 Show in Excel that your Exercise 4.1 solutions solve equation 4.2.

PROBLEM 4.5 Silverman Bags Co. issues the virtual City Ltd. (Exercise 4.2) for £800,000, and hedges the issue by dynamic positions in a traded City Ltd. Index. Indicate the appropriate positions, and likely initial profits in this activity.

PROBLEM 4.6 Philip Hedge Fund ("PHF") buys 10% of City Ltd. at your calculated value per share, and hedges by going short in the City Ltd. Index at 5% cost for each transaction. The next four end week City share prices are £1.00, £.82, £.80 and £.80, V prices are per square foot 800, 750, 700 and 750, and interest rates are 4%, 2%, 5% and 4%. Evaluate PF's performance in terms of risk and return. Does PHF deserve a 20% gain fee for this activity?

Appendix 4A DERIVATIONS

Let F(V) denote the value of the option to invest and let us construct a portfolio, π, which replicates the option's value. This portfolio is assumed not to pay any dividends and therefore its return will only be capital gains. Such a portfolio will be formed by a long position in the option and a short position (delta) on n units of the project. The value of this portfolio is given by:

$$\pi = F(V) - nV \tag{A1}$$

The short position in this portfolio will require a payment of $\delta Vndt$. In a risk neutral world, the parameter δ=r-μ, termed the dividend or payout rate or the convenience yield, represents the difference between the riskless rate of return and the expected growth rate of the project. In a short period of time this portfolio will be worth:

$$dF(V) - ndV - \delta Vndt \tag{A2}$$

The changes in the value of the option, dF(V), depend on the stochastic variable, V, therefore an expansion of its terms can be done using Ito's Lemma:

$$dF(V) = \frac{\partial F}{\partial V} dV + \frac{1}{2} \frac{\partial^2 F}{\partial V^2} dV^2 \tag{A3}$$

Substituting (4.1) into (A3) and into (A2) and recollecting the terms we obtain:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} dt + \alpha V \frac{\partial F}{\partial V} dt - \alpha Vndt + \sigma V \frac{\partial F}{\partial V} dz - \sigma Vndz - \delta Vndt \tag{A4}$$

If $n = \frac{\partial F}{\partial V} = \Delta$ the random terms in equation (A4) disappear yielding:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} dt - \delta V \frac{\partial F}{\partial V} dt \tag{A5}$$

Since there is no randomness in this portfolio, its return should be a risk free return. Thus, denoting r as the risk free rate:

$$\frac{1}{2}\sigma^2 V^2 \frac{\partial^2 F}{\partial V^2} dt - \delta V \frac{\partial F}{\partial V} dt = r\left[F - \frac{\partial F}{\partial V} V \right] dt \tag{A6}$$

Dividing (A6) by dt and recollecting the terms, we obtain the differential equation 4.6.

Guess function solution as $F(V) = A \ \ V^{\beta_1}$

$$\frac{1}{2}\sigma^2 V^2 F''(V) + (r-\delta)VF'(V) - rF(V) = 0$$

$$F'(V) = \beta_1 A V^{\beta_1 - 1} \qquad (A7)$$

$$F''(V) = \beta_1(\beta_1 - 1)AV^{\beta_1 - 2} \qquad (A8)$$

substitute $F''(V)$ *and* $F'(V)$

$$\frac{1}{2}\sigma^2 V^2 \beta_1(\beta_1 - 1)AV^{\beta_1 - 2} + (r-\delta)V\beta_1 AV^{\beta_1 - 1} - rAV^{\beta_1} = 0$$

divide by AV^{β_1}

$$\frac{1}{2}\sigma^2 \beta_1(\beta_1 - 1) + (r-\delta)\beta_1 - r = 0$$

Solve Quadratic

$$f(x) = ax^2 + bx + c$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Some Power Function Rules

$$a^{-n} = \frac{1}{a^n}$$

$$a^n * a^m = a^{n+m}$$

$$\frac{a^n}{a^m} = a^{n-m}$$

$$(a^n)^m = a^{n*m}$$

$$(a*b)^n = a^n * b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^1 = a$$

$$a^0 = 1$$

Substitute function into Value Matching and Smooth Pasting Conditions

$$F(V^*) = AV^{*\beta_1} = V^* - K \quad \textit{Value Match}$$

$$F'(V^*) = A\beta_1 V^{*\beta_1 - 1} = 1 \quad \textit{Smooth Paste}$$

$$A = (V^* - K)/V^{*\beta_1}$$

$$A = 1/[\beta_1 V^{*\beta_1 - 1}]$$

Re *arrange*

$$V^* - K = \frac{V^{*\beta_1}}{\beta_1 V^{*\beta_1 - 1}}$$

$$\beta_1 K = V^*(\beta_1 - 1) \qquad \textit{so}$$

$$V^* = \frac{\beta_1}{\beta_1 - 1} K$$

Since

$$F'(V^*) = 1 = A \quad \beta_1 [\frac{\beta_1}{\beta_1 - 1} K]^{\beta_1 - 1}$$

$$A = [\frac{(\beta_1 - 1)^{\beta_1 - 1}}{\beta_1^{\beta_1} K^{\beta_1 - 1}}]$$